Inequality in a right triangle.

https://www.linkedin.com/feed/update/urn:li:activity:6654694204264054784

Let x and y be legs and h the hypotenuse of a right triangle. Prove that

$$\frac{1}{2h+x+y}+\frac{1}{h+2x+y}+\frac{1}{h+x+2y}<\frac{h}{2xy}.$$

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Assume $h=1$ (due homogeneity of the inequality). Then the inequality becomes

(**1**) $$\frac{1}{2+x+y}+\frac{1}{1+2x+y}+\frac{1}{1+x+2y}\le\frac{1}{2xy}.$$

Since $x^2+y^2=1$ then denoting $p:=x+y, q:=xy\le\frac{1}{2}$ we obtain

$$\frac{1}{2+x+y}+\frac{1}{1+2x+y}+\frac{1}{1+x+2y}=$$

$$\frac{11(x+y)+11xy+5(x^2+y^2)+5}{2(x^3+y^3)+7xy(x+y)+7(x^2+y^2)+16xy+7(x+y)+2}=$$

$$\frac{11(x+y)+11xy+10}{2(x+y)(1-xy)+7xy(x+y)+9+16xy+7(x+y)}=$$

$$\frac{11p+11q+10}{2p(1-q)+7qp+9+16q+7p}=\frac{11p+11q+10}{9p+16q+5pq+9}.$$

Thus, we have $\frac{1}{2xy}-\frac{1}{2+x+y}-\frac{1}{1+2x+y}-\frac{1}{1+x+2y}=$

$\frac{1}{2q}-\frac{11p+11q+10}{9p+16q+5pq+9}=\frac{9+9p-4q-17pq-22q^2}{2q(9p+16q+5pq+9)}>0$ because

$$9+9p-4q-17pq-22q^2\ge 9+9p-4\cdot\frac{1}{2}-17p\cdot\frac{1}{2}-22\cdot\frac{1}{4}=\frac{p}{2}+\frac{3}{2}>0.$$